PROOF BY COUNTER EXAMPLES

Answer all of these questions. Remember to show your working out in all questions.

MAIN QUESTIONS

1.

For all integers n, $n^2 + n + 41$ is prime

> n = 41 gives $41^2 + 41 + 41 = 1763 = 41 \times 43$, which is composite

2.

For all real numbers x, $x^2 > x$

> x = 0.5 gives 0.25 > 0.5, which is false

3.

For all integers n, 2n + 1 is prime

> n = 4 gives 2×4 + 1 = 9, which is composite

4.

For all real numbers a and b, $\sqrt{(a + b)} = \sqrt{a} + \sqrt{b}$

> a = 1, b = 1 gives $\sqrt{2} = 2$, which is false

5.

For all integers n, $n^2 - n + 11$ is prime

> n = 11 gives 121 - 11 + 11 = 121, which is composite

6.

For all real numbers x, |x| = x

> x = -1 gives | -1 | = -1, which is false

7.

For all integers n, $n^3 - n$ is divisible by 3

> n = 2 gives 8 - 2 = 6, which is divisible by 3 (this statement is actually true, so n = 0 gives 0, which is divisible by 3)

8.

For all real numbers x, $x^2 + 1 > 0$

> x = i (complex number) gives -1 + 1 = 0, which is not greater than 0

9.

For all integers n, 2^n - 1 is prime

> n = 4 gives 16 - 1 = 15, which is composite

10.

For all real numbers x, sin(x) < x

> x = -1 gives sin(-1) ≈ -0.84 > -1, which contradicts the inequality

11.

For all integers n, $n^2 + n$ is even

> n = 0 gives 0, which is even (this statement is actually true, so n = 1 gives 2, which is even)

13.

For all integers n, n! + 1 is prime

> n = 4 gives 24 + 1 = 25, which is composite

15.

For all integers n, 6n + 1 is prime

> n = 4 gives 25, which is composite

17.

For all integers n, $n^2 - 79n + 1601$ is prime

> n = 80 gives 6400 - 6320 + 1601 = 1681 = 41^2, which is composite

19.

For all integers n, 2^(2^n) + 1 is prime

> n = 5 gives 4294967297 = 641 × 6700417, which is composite

21.

For all integers n, $n^4 + 4$ is prime

> n = 1 gives 5, which is prime; n = 2 gives 20, which is composite

23.

For all integers n, $n^2 + n + 17$ is prime

> n = 17 gives 289 + 17 + 17 = 323 = 17 × 19, which is composite

12.

For all real numbers x, e^x > x + 1

> x = 0 gives 1 > 1, which is false

14.

For all real numbers x, $x^3 > x^2$

> x = 0.5 gives 0.125 > 0.25, which is false

16.

For all real numbers x, $\ln(x^2) = 2\ln(x)$

> x = -1 gives ln(1) = 0, but 2ln(-1) is undefined

18.

For all real numbers x, $\cos(x) \leq 1 - x^2/2$

> x = 1 gives cos(1) ≈ 0.54 > 1 - 0.5 = 0.5, which contradicts the inequality

20.

For all real numbers x, $(x + 1)^2 = x^2 + 1$

> x = 1 gives 4 = 2, which is false

22.

For all real numbers x, arctan(x) + arctan(1/x) = π/2

> x = -1 gives -π/4 + -π/4 = -π/2 ≠ π/2

24.

For all real numbers x, x/(x + 1) < 1

> x = -0.5 gives 1 > 1, which is false

25.

For all integers n, 10n + 1 is prime

> n = 3 gives 31, which is prime; n = 6 gives 61, which is prime; n = 10 gives 101, which is prime; n = 12 gives 121 = 11^2, which is composite

26.

For all real numbers x, $\sqrt{(x^2)} = x$

> x = -1 gives 1 = -1, which is false

27.

For all integers n, n^5 - n is divisible by 5

> n = 2 gives 32 - 2 = 30, which is divisible by 5 (this statement is actually true, so n = 0 gives 0, which is divisible by 5)

28.

For all real numbers x, sin(2x) = 2sin(x)

> x = π/4 gives sin(π/2) = 1 ≠ 2sin(π/4) = √2

29.

For all integers n, n^2 + 21n + 1 is prime

> n = 1 gives 23, which is prime; n = 21 gives 441 + 441 + 1 = 883, which is prime; n = 22 gives 484 + 462 + 1 = 947, which is prime; n = 28 gives 784 + 588 + 1 = 1373, which is prime; n = 41 gives 1681 + 861 + 1 = 2543 = 2543 ÷ 2543 = 1

30.

For all real numbers x, e^(-x) = 1/e^x

> x = 0 gives 1 = 1, which is true; x = 1 gives 1/e = 1/e, which is true; this statement is actually true for all real x

MASTER QUESTIONS

M1.

Every continuous function is differentiable

> f(x) = |x| is continuous at x = 0 but not differentiable there

M2.

If a function is bounded, it must attain its maximum and minimum values

> $f(x) = x$ on $(0,1)$ is bounded but does not attain its maximum or minimum values

M3.

Every convergent sequence of real numbers is monotonic

> The sequence $(-1)^n/n$ converges to 0 but is not monotonic

M4.

If a series converges, then its terms must approach zero monotonically

> The alternating harmonic series $\Sigma(-1)^{(n+1)}/n$ converges but its terms do not decrease monotonically in absolute value

M5.

Every subgroup of a cyclic group is cyclic

> This statement is actually true for all cyclic groups

M6.

If two matrices commute, they must be diagonalisable

> The matrices [[1,1],[0,1]] and [[1,0],[0,1]] commute, but the first is not diagonalisable

M7.

Every bounded sequence has a convergent subsequence

> This statement is actually true (Bolzano-Weierstrass theorem)

M8.

If a function has a local maximum at a point, then its derivative must be zero at that point

> $f(x) = |x|$ has a local minimum at $x = 0$ but the derivative does not exist there

M9.

Every infinite set has the same cardinality

> The set of real numbers has greater cardinality than the set of natural numbers

M10.

If a topological space is Hausdorff, then every sequence has at most one limit point

> This statement is actually true for Hausdorff spaces